PREPARING FOR COMPLETING THE SQUARE

❑ INTRODUCTION

I assume that the only methods you have learned for solving a *quadratic equation* are the **factoring method (**and perhaps the **Quadratic Formula)**. For example, to solve the quadratic equation

$$2x^2 + 3x - 20 = 0$$

by factoring, you would proceed as follows:

$$
\begin{array}{lll}
 & 2x^2 + 3x - 20 = 0 & \text{(the given equation)} \\
\Rightarrow & (2x - 5)(x + 4) = 0 & \text{(factor the quadratic)} \\
\Rightarrow & 2x - 5 = 0 \quad \text{OR} \quad x + 4 = 0 & \text{(set each factor to 0)} \\
\Rightarrow & x = \frac{5}{2} \quad \text{OR} \quad x = -4\text{, and we're done — two solutions.} &
\end{array}
$$

So factoring is a good enough method, you say? I cannot agree with you; I submit to you the following quadratic equation:

$$x^2 + 7x + 5 = 0$$

It's not a complicated equation — the numbers are small, and there aren't even any negative numbers to mess with. So go ahead, I dare you to factor that thing.

See my point? Trust me, that equation <u>does</u> have two solutions, but factoring is not the method that will yield those two solutions. We need a new method for solving quadratic equations, and this chapter will give you some of the prerequisite skills required for a future chapter that covers the technique called *Completing the Square*.

❑ *FACTORING PERFECT SQUARE TRINOMIALS*

Let's review the term ***perfect square***. The number 100 is a perfect square because 100 can be written as the square of 10: $100 = 10^2$. Also, the expression $(x + 5)^2$ is a perfect square, since it is the square of $x + 5$. Even n^6 is a perfect square because it is the square of n^3: $n^6 = (n^3)^2$.

One of the steps in solving a quadratic equation by *completing the square* is factoring a ***perfect square trinomial***. Let's look at four examples.

<u>Example 1:</u>

$x^2 + 14x + 49$ is a perfect square trinomial because it's a trinomial that factors into the square of a binomial:

$$x^2 + 14x + 49 = (x + 7)(x + 7) = (x + 7)^2$$

[perfect square trinomial] [square of a binomial]

<u>Example 2:</u>

$n^2 - 20n + 100$ is a perfect square trinomial:

$$n^2 - 20n + 100 = (n - 10)(n - 10) = (n - 10)^2$$

Here are a couple of examples with fractions:

<u>Example 3:</u>

$$a^2 + 3a + \frac{9}{4} = \left(a + \frac{3}{2}\right)\left(a + \frac{3}{2}\right) = \left(a + \frac{3}{2}\right)^2$$

Check:

$$\left(a + \frac{3}{2}\right)^2 = \left(a + \frac{3}{2}\right)\left(a + \frac{3}{2}\right) = a^2 + \underbrace{\frac{3}{2}a + \frac{3}{2}a}_{\frac{3}{2}+\frac{3}{2}=\frac{6}{2}=3} + \frac{9}{4} = a^2 + 3a + \frac{9}{4} \checkmark$$

Example 4:

$$y^2 - \frac{2}{5}y + \frac{1}{25} = \left(y - \frac{1}{5}\right)\left(y - \frac{1}{5}\right) = \left(y - \frac{1}{5}\right)^2$$

Check:

$$\left(y - \frac{1}{5}\right)\left(y - \frac{1}{5}\right) = y^2 - \frac{1}{5}y - \frac{1}{5}y + \frac{1}{25} = y^2 - \frac{2}{5}y + \frac{1}{25} \checkmark$$

Homework

1. Factor each perfect square trinomial:

a. $x^2 + 10x + 25$

b. $y^2 - 18y + 81$

c. $a^2 + a + \frac{1}{4}$

d. $m^2 - \frac{4}{3}m + \frac{4}{9}$

e. $z^2 + \frac{2}{5}z + \frac{1}{25}$

f. $w^2 - \frac{5}{3}w + \frac{25}{36}$

g. $b^2 + \frac{9}{5}b + \frac{81}{100}$

h. $u^2 + \frac{3}{2}u + \frac{9}{16}$

i. $n^2 - \frac{4}{7}n + \frac{4}{49}$

j. $x^2 + \frac{10}{11}x + \frac{25}{121}$

❑ *REVIEW OF SOLVING QUADRATICS BY TAKING SQUARE ROOTS*

Another skill required for completing the square is taking square roots to solve quadratic equations. You might want to review *Quadratic Equations by Taking Square Roots*. Here's an example from that chapter.

EXAMPLE 5: **Solve the quadratic equation: $(x + 7)^2 = 81$**

Solution: According to the Square Root Theorem, we can remove the squaring by taking the square root of both sides of the equation, remembering that the number 81 has two square roots:

Remember!

81 has two square roots.

$$(x+7)^2 = 81 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad x+7 = \pm\sqrt{81} \qquad \text{(the Square Root Theorem)}$$

$$\Rightarrow \quad x+7 = \pm 9 \qquad (\sqrt{81} = 9)$$

$$\Rightarrow \quad x = -7 \pm 9 \qquad \text{(subtract 7 from each side)}$$

Using the plus sign yields $x = -7 + 9 = 2$.

Using the minus sign yields $x = -7 - 9 = -16$.

$$\boxed{x = 2 \text{ or } -16}$$

❑ THE "MAGIC NUMBER"

Consider the trinomial $x^2 + 10x + 25$. We've learned that its factorization is

$$x^2 + 10x + 25 = (x+5)^2, \text{ which is the square of a binomial.}$$

Let's look carefully at the numbers in this equality. Notice that the 5 is half of the 10, and that the 25 is the square of the 5.

Let's do one more example. Consider the factorization

$$n^2 - 14n + 49 = (n-7)^2, \text{ which is the square of a binomial.}$$

We note that the -7 is half of the -14, and that the 49 is the square of the -7.

So now imagine that I give you

$$x^2 + 6x$$

and I ask you to add a third term to this binomial so that the resulting trinomial will factor into the square of a binomial:

$$x^2 + 6x + \underline{???} \;=\; (x + ?)(x + ?) \;=\; (x + ?)^2$$

Each single "?" must be 3, since 3 is half of 6. Also, the "???" must be 9, since 9 is the square of the 3. In other words,

$$x^2 + 6x + \boxed{9} \;=\; (x + \underline{\mathbf{3}})(x + \underline{\mathbf{3}}) \;=\; (x + \underline{\mathbf{3}})^2$$

We shall call 9 the "magic number." It can be calculated for this problem using the following two-step rule:

1) Calculate <u>half</u> of 6, which is 3.
2) <u>Square</u> the 3, which is **9**, the "magic number."

Note: When we convert $x^2 + 6x$ to $x^2 + 6x + 9$ by adding the "magic number" **9**, we are <u>not</u> saying that they're equal, but there will always be a way of adding the magic number without violating any of the laws of algebra.

To become proficient in completing the square, we must be really good at finding the *magic number*. The following chart gives more examples of how this is done. We'll let b represent the number in front of the variable (the coefficient of the linear term).

Original Quadratic	Value of *b*	<u>Half</u> of *b*	Half of *b* <u>Squared</u> = **The Magic Number**	New Quadratic	New Quadratic in Factored Form
$x^2 + 22x$	22	11	$\mathbf{121}$	$x^2 + 22x + \mathbf{121}$	$(x + 11)^2$
$A^2 - 14A$	-14	-7	$\mathbf{49}$	$A^2 - 14A + \mathbf{49}$	$(A - 7)^2$
$n^2 + 9n$	9	$\frac{9}{2}$	$\mathbf{\frac{81}{4}}$	$x^2 + 9x + \mathbf{\frac{81}{4}}$	$\left(x + \frac{9}{2}\right)^2$
$y^2 - \frac{2}{5}y$	$-\frac{2}{5}$	$-\frac{1}{5}$	$\mathbf{\frac{1}{25}}$	$y^2 - \frac{2}{5}y + \mathbf{\frac{1}{25}}$	$\left(y - \frac{1}{5}\right)^2$
$3u^2 - 7u$	This problem can't be done yet; the leading coefficient is <u>not</u> 1.				

Homework

2. Find the **magic number** for each quadratic binomial:

a. $a^2 + 8a$ b. $x^2 - 12x$ c. $y^2 + 28y$

d. $b^2 + 3b$ e. $c^2 - 7c$ f. $t^2 + 11t$

g. $x^2 + x$ h. $y^2 - y$ i. $z^2 + 2z$

j. $g^2 + \frac{2}{3}g$ k. $a^2 - \frac{5}{7}a$ l. $x^2 - \frac{7}{13}x$

Review Problems

3. How many solutions does each quadratic equation have?

a. $(x + 3)(x + 4) = 0$ ___ b. $(x + 9)^2 = 0$ ___

c. $(x - 1)^2 = 10$ ___ d. $(x + 6)^2 = -9$ ___

4. Factor each trinomial:

a. $n^2 + 10n + 25$ b. $x^2 - 18x + 81$

c. $a^2 + 5a + \frac{25}{4}$ d. $x^2 - \frac{4}{3}x + \frac{4}{9}$

e. $y^2 + \frac{6}{7}y + \frac{9}{49}$ f. $t^2 - \frac{1}{5}t + \frac{1}{100}$

5. Find the magic number for $w^2 - \frac{7}{9}w$.

Solutions

1. a. $(x+5)^2$ b. $(y-9)^2$ c. $\left(a+\frac{1}{2}\right)^2$ d. $\left(m-\frac{2}{3}\right)^2$

e. $\left(z+\frac{1}{5}\right)^2$ f. $\left(w-\frac{5}{6}\right)^2$ g. $\left(b+\frac{9}{10}\right)^2$ h. $\left(u+\frac{3}{4}\right)^2$

i. $\left(n-\frac{2}{7}\right)^2$ j. $\left(x+\frac{5}{11}\right)^2$

2. a. $\frac{1}{2}\cdot 8 = 4$, and $4^2 = \mathbf{16}$

b. $\frac{1}{2}\cdot -12 = -6$, and $(-6)^2 = \mathbf{36}$

c. $\frac{1}{2}\cdot 28 = 14$, and $14^2 = \mathbf{196}$

d. $\frac{1}{2}\cdot 3 = \frac{3}{2}$, and $\left(\frac{3}{2}\right)^2 = \mathbf{\frac{9}{4}}$

e. $\frac{1}{2}\cdot -7 = -\frac{7}{2}$, and $\left(-\frac{7}{2}\right)^2 = \mathbf{\frac{49}{4}}$

f. $\frac{1}{2}\cdot 11 = \frac{11}{2}$, and $\left(\frac{11}{2}\right)^2 = \mathbf{\frac{121}{4}}$

g. $\frac{1}{2}\cdot 1 = \frac{1}{2}$, and $\left(\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$

h. $\frac{1}{2}\cdot -1 = -\frac{1}{2}$, and $\left(-\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$

i. $\frac{1}{2}\cdot 2 = 1$, and $1^2 = \mathbf{1}$

j. $\frac{1}{2}\cdot \frac{2}{3} = \frac{1}{3}$, and $\left(\frac{1}{3}\right)^2 = \mathbf{\frac{1}{9}}$

k. $\frac{1}{2}\cdot -\frac{5}{7} = -\frac{5}{14}$, and $\left(-\frac{5}{14}\right)^2 = \mathbf{\frac{25}{196}}$

l. $\frac{1}{2}\cdot -\frac{7}{13} = -\frac{7}{26}$, and $\left(-\frac{7}{26}\right)^2 = \mathbf{\frac{49}{676}}$

3. a. 2 b. 1 c. 2 d. 0

4. a. $(n + 5)^2$ b. $(x - 9)^2$ c. $\left(a + \frac{5}{2}\right)^2$

d. $\left(x - \frac{2}{3}\right)^2$ e. $\left(y + \frac{3}{7}\right)^2$ f. $\left(t - \frac{1}{10}\right)^2$

5. $\frac{49}{324}$

Education is not
a preparation for life;
education is life itself.

John Dewey